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V–A sum rules with $D = 10$ operators

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Abstract. The difference of vector and axial-vector charged current correlators is analyzed by means of QCD sum rules. The contribution of 10-dimensional 4-quark condensates is calculated and its value is estimated within the framework of the factorization hypothesis. It is compared to the result obtained from an operator fit of Borel sum rules in the complex q^2-plane, calculated from experimental data on hadronic τ-decays. This fit gives accurate values of the light quark condensate and the quark–gluon mixed condensate. The size of the high-order operators and the convergence of the operator series are discussed.

1 Introduction

The QCD sum rules [1] have been widely used for the determination of the fundamental theoretical parameters, such as the coupling constant α_s, quark masses and various non-perturbative condensates. Their accuracy depends on experimental errors and theoretical uncertainties. In many cases both experimental and theoretical errors are comparable by the order of magnitude, and any improvement is of interest.

In this paper we will consider the 2-point correlators of charged vector and axial-vector currents, constructed from light u, d-quarks:

$$\Pi^U_{\mu\nu}(q) = \mathrm{i}\int \mathrm{d}x\, \mathrm{e}^{\mathrm{i}qx} \left\langle T U^\dagger_\mu(x) U_\nu(0)\right\rangle \tag{1}$$
$$= (q_\mu q_\nu - g_{\mu\nu} q^2)\, \Pi^{(1)}_U(q^2) + q_\mu q_\nu\, \Pi^{(0)}_U(q^2),$$

where

$$U = V, A: \qquad V_\mu = \bar u \gamma_\mu d\,,\ A_\mu = \bar u \gamma_\mu \gamma^5 d\,.$$

The polarization functions $\Pi^{(i)}(s)$ have a cut along the real axes in the complex $s = q^2$-plane. Their imaginary parts (spectral functions),

$$v_1/a_1(s) = 2\pi\, \mathrm{Im}\, \Pi^{(1)}_{V/A}(s + \mathrm{i}0)\,,$$
$$a_0(s) = 2\pi\, \mathrm{Im}\, \Pi^{(0)}_A(s + \mathrm{i}0), \tag{2}$$

have been measured for $0 < s < m_\tau^2$ by the ALEPH [2] and OPAL [3] collaborations from hadronic decays of the τ-lepton.

Of particular interest is the difference $\Pi^{(1)}_V - \Pi^{(1)}_A$, since it does not contain any perturbative contribution in the massless quark limit. The experimental data on the difference $v_1(s) - a_1(s)$ are shown in Fig. 1. As demonstrated

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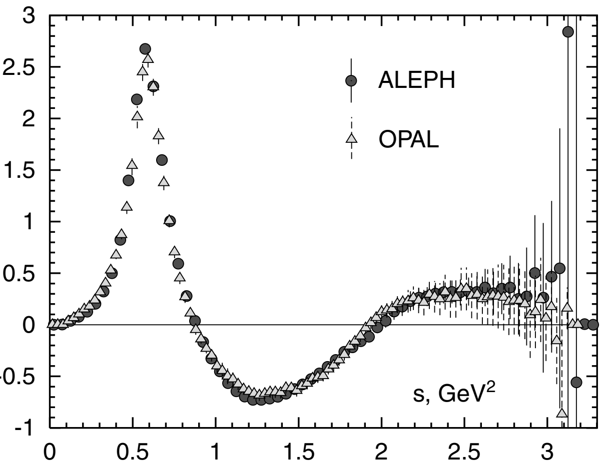

Fig. 1. Spectral function $v_1(s) - a_1(s)$ obtained from ALEPH [2] and OPAL [3] data

in [4], the dispersion relation can be written in the following form:

$$\Pi^{(1)}_V(s) - \Pi^{(1)}_A(s) = \frac{1}{2\pi^2}\int_0^\infty \frac{v_1(t) - a_1(t)}{t - s}\,\mathrm{d}t + \frac{f_\pi^2}{s}$$
$$= \sum_{D\geq 4} \frac{O_D^{V-A}}{(-s)^{D/2}}, \tag{3}$$

where the sum goes over even dimensions D of the operators (condensates) O_D. The term $\frac{f_\pi^2}{s}$ ($f_\pi = 130.7\,\mathrm{MeV}$ is the pion decay constant) is the kinematical pole of the axial polarization function $\Pi^A_{\mu\nu}$; see [4] for details. In (3) and below the notation O_D^{V-A} stands for the condensates with all α_s corrections, including slowly varying logarithmic terms $\sim \ln^n(-s)$. The list of the condensate contributions to the vector and axial correlators separately can be found in [5].

The sum rules for the difference (3) have been studied in [4,6–12] where the lowest order condensates O_D^{V-A} were found. Although the published values of O_6^{V-A} are close to each other (within the errors), this is not the case for the operator O_8^{V-A}. In [4,6] positive values of the $D = 8$ condensate were found, but the authors of recent publications [9,12] have obtained a negative condensate O_8^{V-A}. The source of this discrepancy could be very large condensates of dimension $D = 10$ and higher, accounted for in [9,12]: a typical ratio of the condensates in these papers is $|O_{2n+2}/O_{2n}| \sim 5$–$10\,\mathrm{GeV}^2$. If this statement is correct, the OPE analysis of [4] would be invalid, because the contribution of unknown high-order terms was estimated from the assumption $|O_{10}/O_6| \lesssim 1$–$2\,\mathrm{GeV}^4$. For this reason it would be interesting to find the operator O_{10}^{V-A} independently and compare it with the sum rule results.

In this paper we repeat the analysis of [4] with the $D = 10$ operator included. In Sect. 2 all necessary V–A operators, obtained from the operator product expansion in QCD, are listed and their values are estimated within the framework of the factorization hypothesis. In Sect. 3 the operator values are obtained from the fit to the Borel sum rules. In the last section the validity of our assumptions is discussed and the results are compared with the ones obtained in other publications. The complete form of the $D = 10$ operator and technical details of its derivation are postponed to Appendices A and B.

2 V–A operator expansion

The first term in the operator series (3) is the $D = 4$ operator:

$$O_4^{V-A} = 2(m_u + m_d)\langle\bar{q}q\rangle \times \left[1 + \frac{4}{3}\frac{\alpha_s(Q^2)}{\pi} + \frac{59}{6}\left(\frac{\alpha_s(Q^2)}{\pi}\right)^2\right], \tag{4}$$

where $Q^2 = -q^2$, and we assume $\langle\bar{u}u\rangle = \langle\bar{d}d\rangle \equiv \langle\bar{q}q\rangle$. The α_s corrections have been computed in [13,14]. In fact, the contribution of the $D = 4$ operator to the sum rules considered here is small. So we can safely neglect the α_s corrections in (4) and put $O_4^{V-A} = -f_\pi^2 m_\pi^2 = -3.3 \times 10^{-4}\,\mathrm{GeV}^4$, as follows from the Gell-Mann–Oakes–Renner low energy theorem [15].

The $D = 6$ operator in factorized form is equal to

$$O_6^{V-A} = -8\pi C_N \alpha_s \langle\bar{q}q\rangle^2 \left[1 + \frac{\alpha_s(\mu^2)}{\pi}\left(c_6 - \frac{1}{4}\ln\frac{Q^2}{\mu^2}\right)\right], \tag{5}$$

where $C_N = 1 - N_c^{-2} = 8/9$ is the color factor, which appears in the factorization of the 4-quark operators at the leading α_s order. The NLO terms were computed in [16] and the constant c_6 was found to be equal to 247/48. In [17] another treatment of the γ^5 matrix in dimensional regularization was employed, leading to $c_6 = 89/48$. For the latter choice at $\mu = 1\,\mathrm{GeV}$ and $\alpha_s(\mu^2) = 0.5$ one finds the factor in square brackets in (5) to be equal to 1.3 (the logarithmic term can be neglected due to the small numerical coefficient).

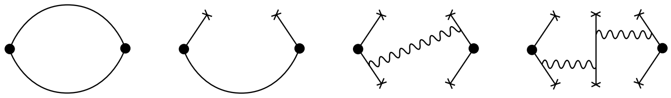

Fig. 2. Condensate expansion (8) by the number of quarks in vacuum. Circles stand for the currents V/A, crosses are quarks in vacuum; gluons in vacuum are not shown

The contribution of the $D = 8$ 4-quark condensates to the vector current correlator was originally obtained in [18] in factorized form and in [19] in complete (non-factorized) form. In [4] these results were verified and an ambiguity of the factorization at the N_c^{-2} order was pointed out. Here we will follow the factorization procedure, described in Appendix B. The result is[1]

$$O_8^{V-A} = 8\pi C_N \alpha_s m_0^2 \langle\bar{q}q\rangle^2, \tag{6}$$

where the mass m_0 is defined from the 5-dimensional quark–gluon mixed condensate:

$$\mathrm{i}\langle\bar{q}\hat{G}q\rangle = 2\langle\bar{q}D^2q\rangle = -m_0^2\langle\bar{q}q\rangle, \tag{7}$$

where $\hat{G} = \gamma_\alpha\gamma_\beta G_{\alpha\beta}$, $G_{\alpha\beta} = \mathrm{i}[D_\alpha, D_\beta]$ is the gluon field strength; see Appendix A for more definitions. The parameter m_0^2 has the meaning of a typical momentum of virtual quarks in vacuum. It was found from baryonic sum rules that $m_0^2 = 0.8 \pm 0.2\,\mathrm{GeV}^2$ [20,21], and also the B–B^* splitting was found [22]. The values close to $1\,\mathrm{GeV}^2$ were also obtained from the latest lattice calculation [23] and in the QCD string model [24].

There are many different condensates of dimension $D = 10$. They can be grouped into four parts:

$$O_{10}^{V-A} = O_{10}^{(0)} + O_{10}^{(2)} + O_{10}^{(4)} + O_{10}^{(6)}, \tag{8}$$

where the upper index (i) denotes the number of quarks in vacuum. This separation is shown diagrammatically in Fig. 2. The purely gluonic operators $O^{(0)}$ and the 2-quark ones $O^{(2)}$ cancel in the V–A correlator in the limit of massless u, d-quarks. The operators with six quarks in vacuum have the structure $\langle(\bar{q}q)^2(\bar{q}Dq)\rangle$. After factorization they become $\sim m\langle\bar{q}q\rangle^3$, which is again negligible for light quarks. The only essential contribution to the V–A sum rules comes from the 4-quark operators $O_{10}^{(4)}$.

In this paper we have computed the contribution of the 4-quark condensates to the vector and axial current correlator. Details of the calculation and the complete form of the operator $O_{10}^{(4)}$ are given in Appendix A. The factorization scheme necessary to reduce the large number of independent structures is described in Appendix B. The result is

$$O_{10}^{V-A} = \pi\alpha_s C_N \times \left[\frac{50}{9}\langle\bar{q}\hat{G}q\rangle^2 - 16\left(3X_1 - X_2 + X_3 + \frac{7}{6}X_4\right)\langle\bar{q}q\rangle\right], \tag{9}$$

[1] In [4] the factor C_N was ignored, since $O(N_c^{-2})$ terms were neglected.

where X_i are four independent $D = 7$ quark–gluon condensates:

$$X_1 = \langle \bar{q} G_{\alpha\beta} G_{\alpha\beta} q \rangle, \quad X_2 = \mathrm{i}\langle \bar{q}\gamma^5 G_{\alpha\beta}\tilde{G}_{\alpha\beta} q\rangle,$$
$$X_3 = \langle \bar{q}\gamma_{\alpha\beta} G_{\alpha\gamma} G_{\beta\gamma} q \rangle, \quad X_4 = \mathrm{i}\langle \bar{q}\gamma_{\alpha\beta}(D_\alpha J_\beta) q\rangle, \tag{10}$$

where $\tilde{G}_{\alpha\beta}$ is the dual gluon field strength, $\gamma_{\alpha\beta} = \frac{1}{2}(\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha)$ and $J_\alpha = D_\beta G_{\alpha\beta}$. Their numerical values are not known. The condensate X_4 can be brought to the 4-quark form $X_4 \sim \langle(\bar{q}q)(\bar{q}Dq)\rangle \sim m\langle\bar{q}q\rangle^2$, which is negligible. In order to estimate other condensates, we assume further factorization according to $\langle\bar{q}\Gamma q\rangle = \langle\bar{q}q\rangle\,\langle\mathrm{tr}\,\Gamma\rangle/(4N_c)$; the trace is taken both over color and spinor indices. Then

$$X_1 = \frac{2\pi}{3}\,\alpha_s\left\langle G^a_{\alpha\beta}G^a_{\alpha\beta}\right\rangle\langle\bar{q}q\rangle\,, \qquad X_2 = X_3 = X_4 = 0\,. \tag{11}$$

Under these assumptions the operator (9) takes the form

$$O_{10}^{V-A} = -\pi\alpha_s C_N\langle\bar{q}q\rangle^2\left[\frac{50}{9}\,m_0^4 + 32\pi\alpha_s\left\langle G^2\right\rangle\right]. \tag{12}$$

It is rather difficult to find an accurate value of the gluon condensate from any sum rule. The detailed analysis of the charmonium sum rules performed in [25] has led to the restriction $\langle\frac{\alpha_s}{\pi}G^2\rangle = 0.009 \pm 0.007\,\mathrm{GeV}^4$, in agreement with many previous estimations. Taking this central value and $m_0^2 = 1\,\mathrm{GeV}^2$, one obtains $O_{10}^{V-A}/O_6^{V-A} = 0.8\,\mathrm{GeV}^4$. For $O_6^{V-A} = -(6.8 \pm 2.1)\times 10^{-3}\,\mathrm{GeV}^6$ [4] we find the following estimation of the $D = 10$ V–A condensate:

$$O_{10}^{V-A} = -5\times 10^{-3}\,\mathrm{GeV}^{10}. \tag{13}$$

In the next section we will compare this estimation with results of the fit, obtained from the sum rules.

3 V–A sum rules

Many different sum rules have been investigated in order to determine the numerical values of the condensates. Most of the authors employ polynomial sum rules: the correlator $\Pi_V^{(1)}(s) - \Pi_A^{(1)}(s)$ is multiplied on some polynomial of s and then integrated over the circle $|s| = s_0$ in the complex s-plane. The advantages are
(1) one does not need to know the spectral function $v_1(s) - a_1(s)$ for $s > s_0$, which allows one to reduce the high error from the region $s \approx m_\tau^2$ by choosing s_0 reasonably below m_τ^2, and
(2) all operators of dimension higher than the polynomial dimension do not enter these sum rules due to the Cauchy theorem. But the disadvantages are also obvious. If the operator expansion (3) is divergent (asymptotic), the Cauchy theorem is not applicable to this series. Moreover, possible logarithmical terms $\sim \ln^k Q^2/Q^{2n}$ appear at the NLO in the α_s expansion. These terms contribute to any polynomial sum rules. It makes uncontrollable the contribution of the high-order operators to the polynomial sum rules at $s_0 \lesssim 2\,\mathrm{GeV}^2$, especially for large ones as obtained in [9,12].

For these reasons we prefer Borel sum rules, where the high-order operators are suppressed as $O_{2n}/n!$. In order to separate out the contributions of different operators from each other, one may consider the Borel transformation in the complex plane of the Borel mass $M^2 \to M^2\mathrm{e}^{\mathrm{i}(\pi-\phi)}$ (which is equivalent to the Borel operator applied to the dispersion relation (3) written along the ray $s \to s\mathrm{e}^{\mathrm{i}\phi}$ in the complex s-plane [4]). The real and imaginary parts of the Borel transformation are

$$\int_0^{s_m}\exp\left(\frac{s}{M^2}\cos\phi\right)\cos\left(\frac{s}{M^2}\sin\phi\right)\times(v_1 - a_1)(s)\,\frac{\mathrm{d}s}{2\pi^2} = f_\pi^2 + \sum_{k=1}^{\infty}(-)^k\,\frac{\cos(k\phi)O_{2k+2}^{V-A}}{k!M^{2k}}\,, \tag{14}$$

$$\int_0^{s_m}\exp\left(\frac{s}{M^2}\cos\phi\right)\sin\left(\frac{s}{M^2}\sin\phi\right)\times(v_1 - a_1)(s)\,\frac{\mathrm{d}s}{2\pi^2M^2} = \sum_{k=1}^{\infty}(-)^k\,\frac{\sin(k\phi)\,O_{2k+2}^{V-A}}{k!M^{2k+2}}\,. \tag{15}$$

We made the imaginary part (15) dimensionless, while the real part (14) has dimension GeV^2 in order to separate out the leading constant term f_π^2. The logarithmical terms are neglected in the RHS of (14) and (15), otherwise the terms $\sim \ln M^2$ appear. The only known logarithmical term is in the α_s correction to the $D = 6$ operator (5). It can be easily taken into account (see [9] for explicit formulae), but its relative contribution is negligible due to the small numerical factor, so we shall ignore it.

The derivation of (14) and (15) from the dispersion relation (3) implies an infinite upper integration limit $s_m = \infty$. Experimental data on the axial function $a_1(s)$ are available only for $s < m_\tau^2 = 3.16\,\mathrm{GeV}^2$. However, the data at $s > 3\,\mathrm{GeV}^2$ are rather unstable and have a large error because of low statistics; see Fig. 1. For this reason we put $s_m = 3.0\,\mathrm{GeV}^2$ in (14) and (15). Removal of the data above this point does not change the Borel transform significantly (if M^2 is not sufficiently large), but may reduce the errors. In fact, the sum rules considered here do not rely on the high-energy data: say, if the upper integration limit s_m is reduced to $2.5\,\mathrm{GeV}^2$, the condensates change at most within the 10% limit. If the data above $3\,\mathrm{GeV}^2$ are removed, both ALEPH and OPAL data give almost equal central values and similar errors of the Borel transforms (14) and (15). For this reason we will present below the analysis of the ALEPH data only, since they have smaller errors. The condensates obtained from the OPAL data are almost the same.

The argument of the exponent must be negative, $\cos\phi < 0$, in order to suppress the contribution of the high-energy states from the unknown region $s > m_\tau^2$, which means $\pi/2 < \phi < \pi$. Of special interest are the angles closest to π (minimal error), at which the contribution of some operator

Table 1. Operator fit obtained from (15) for different angles ϕ; the operators are in $10^{-3}\,\mathrm{GeV}^D$. The last two lines contain combined fits for all these angles for the upper/lower choice of the M^2 range

ϕ	M^2, GeV^2	O_6^{V-A}	O_8^{V-A}	O_{10}^{V-A}	χ_0^2
$2\pi/3$	0.4–1.0	-7.0 ± 1.4		-3.8 ± 3.3	0.14
	0.5–1.0	-7.0 ± 2.5		-3.9 ± 10.3	0.14
$3\pi/4$	0.4–1.0	-7.3 ± 1.1	8.0 ± 2.1		0.17
	0.5–1.0	-7.9 ± 2.3	9.5 ± 2.5		0.12
$4\pi/5$	0.3–1.0	-8.1 ± 1.8	10.3 ± 4.1	-7.6 ± 5.0	0.11
	0.4–1.0	-9.3 ± 5.1	13.8 ± 15.1	-13.2 ± 23.8	0.06
all		-7.2 ± 1.2	7.8 ± 2.5	-4.4 ± 2.8	0.40
		-7.5 ± 2.3	8.6 ± 6.0	-5.2 ± 8.4	0.20

O_{2k+2} vanishes. Such angles are $\phi = \pi(2k-1)/(2k)$, $k = 2, 3, \ldots$ for the real part (14) and $\phi = \pi(k-1)/k$, $k = 3, 4, \ldots$ for the imaginary one (15). The sum rules (14) and (15) at some of these angles were considered in [4], with the operators O_6 and O_8 as free parameters to fit. It was shown that for $O_6^{V-A} = -(6.8 \pm 2.1) \times 10^{-3}\,\mathrm{GeV}^6$ and $O_8^{V-A} = (7 \pm 4) \times 10^{-3}\,\mathrm{GeV}^6$ they are well satisfied for $M^2 > 0.6\,\mathrm{GeV}^2$.

It is more difficult to find high-order condensates (say O_{10}^{V-A}) from the sum rules, since several unknown parameters enter the same equation and the high-order condensate strongly depends on the exact values of the low-order ones. Here one needs to consider the Borel transformation at several values of M^2, where the relative contributions of various condensates are different. In other words, we may fit the shape of the theoretical curve with an experimental one within some reasonable region $M_1^2 < M^2 < M_2^2$. For this purpose it is natural to define the least square deviation, normalized to experimental error:

$$\chi^2 = \frac{1}{M_2^2 - M_1^2} \int_{M_1^2}^{M_2^2} \mathrm{d}M^2 \left(\frac{B^{\mathrm{theor}} - B^{\mathrm{exp}}}{\Delta B^{\mathrm{exp}}} \right)^2 \tag{16}$$

where $B^{\mathrm{theor}}/B^{\mathrm{exp}}$ is the right/left hand side of the Borel sum rules (14) and (15). One may calculate χ^2 with the theoretical condensates O_i as free parameters. It is a quadratic function of them:

$$\chi^2 = \chi_0^2 + \sum_{i,j} C_{ij} \left(O_i - \overline{O}_i\right) \left(O_j - \overline{O}_j\right). \tag{17}$$

Obviously $\overline{O}_i$ are the central values of the condensates. According to the definition (16) it is natural to consider the equation $\chi^2 = 1$ as the one which determines the border of the 1σ deviation area in the parameter space. Diagonalizing the matrix C_{ij} by means of an orthogonal rotation we conclude that C_{ij} is inverse to the covariance matrix $\overline{\Delta O_i \cdot \Delta O_j} = (C^{-1})_{ij}$. For a good fit $\chi_0^2 \ll 1$.

The fit results depend on the Borel mass limits $M_{1,2}^2$ in (16). For $M^2 > 1\,\mathrm{GeV}^2$ the experimental errors are large, so we take $M_2^2 = 1\,\mathrm{GeV}^2$. The lower limit M_1^2 depends on the size of the neglected high-order operators. In [4] a good coincidence of experimental and theoretical curves was observed for $M^2 > 0.6\,\mathrm{GeV}^2$. Here we include the operator O_{10} in the analysis, so this value can be slightly reduced. As follows from our calculation of the 4-quark condensates (5), (6) and (12), it is reasonable to assume $O_{2n+2}/O_{2n} \sim m_0^2 \approx 0.7\,\mathrm{GeV}^2$. It leads to an estimation $|O_{12}^{V-A}| \sim 3 \times 10^{-3}\,\mathrm{GeV}^{-12}$, which allows us to take $M_1^2 = 0.4\,\mathrm{GeV}^2$, where a typical contribution of such an operator is not higher than 20%. At the angles where the contribution of the operator O_{12} vanishes, the Borel mass can be reduced even further, say, to $M_1^2 = 0.3\,\mathrm{GeV}^2$. All these assumptions are confirmed by the results of the fit; see the figures below.

The condensates, obtained from real part of Borel transformation (14) are sensitive to the exact value of f_π. For this reason we shall use the imaginary part (15) for the numerical fit. The best angles are $\phi = 2\pi/3, 3\pi/4, 4\pi/5$ where the contribution of the operators O_8, O_{10}, O_{12} vanishes respectively. The fit results for each angle are summarized in Table 1. The lowest errors are obtained from the 2-parameter fits at the first two angles. The deviation χ_0^2 for these fits is sufficiently small. For this reason the inclusion of additional parameters, say O_{12}, will not improve the fit quality, but will increase the errors only.

The operator values, obtained from the sum rules, are not independent but have large covariances:

$$\rho_{ij} = \overline{\Delta O_i\, \Delta O_j} / (\overline{(\Delta O_i)^2}\ \overline{(\Delta O_j)^2})^{1/2}.$$

All fits give $\rho_{6,10} \approx 1$ and $\rho_{6,8} \approx \rho_{8,10} \approx -1$. For the 2-parameter fits the covariances can be demonstrated on the confidence level plots; see Fig. 3. The equations $\chi^2 = n^2$ set the ellipses which are the borders of the $n\sigma$ deviation area.

One may also try to fit the condensates at all these angles simultaneously by minimizing $\chi^2_{\mathrm{all}} = \frac{1}{3}[\chi^2(2\pi/3) + \chi^2(3\pi/4) + \chi^2(4\pi/5)]$; see the last two lines in the table. As the final result of our analysis we take this combined fit:

$$\begin{aligned} O_6^{V-A} &= -\,(7.2 \pm 1.2) \times 10^{-3}\,\mathrm{GeV}^6, \\ O_8^{V-A} &= (7.8 \pm 2.5) \times 10^{-3}\,\mathrm{GeV}^8, \\ O_{10}^{V-A} &= -\,(4.4 \pm 2.8) \times 10^{-3}\,\mathrm{GeV}^{10}. \end{aligned} \tag{18}$$

The lower limit of the Borel mass in (16) was taken as $M_1^2 = 0.4\,\mathrm{GeV}^2$ for the first two angles and $M_1^2 = 0.3\,\mathrm{GeV}^2$ for the last one. If M_1^2 is taken by $0.1\,\mathrm{GeV}^2$ higher, the errors

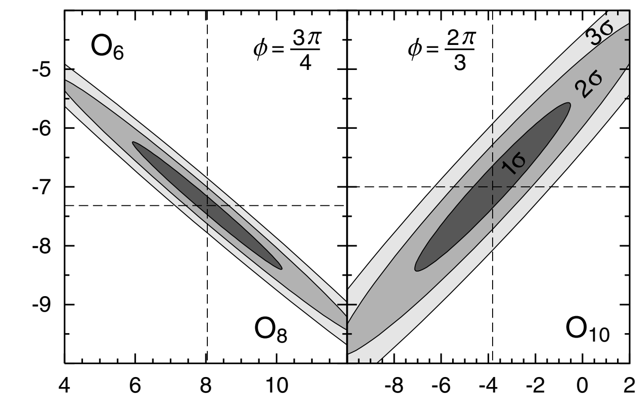

Fig. 3. Confidence level contours, obtained from 2-parameter fits of the sum rule (15) in the range $M^2 = 0.4$–$1\,\mathrm{GeV}^2$. The condensates are in $10^{-3}\,\mathrm{GeV}^D$, the contours show $1, 2, 3\sigma$ deviations of χ^2

are increased, especially for the high dimension operators; see the last line in the table.

The validity of our assumptions is demonstrated in Fig. 4. If the operator O_{10}^{V-A} is taken into account, a good agreement of theoretical and experimental values is observed for $M^2 > 0.4\,\mathrm{GeV}^2$. Below this value the contribution of the operator O_{12} could be large. Even better agreement can be found at the angles where the operator O_{12} disappears; see the plots in Figs. 5. Here the fit can be extended down to $M^2 = 0.3\,\mathrm{GeV}^2$. One may also obtain the condensates by fitting the real part of the Borel transformation (14). Here the central values of the condensates turns out to be close to (18), but the errors are higher due to the presence of the additional parameter f_π^2. The combined fit of (14) at different angles ϕ gives $f_\pi = 131 \pm 4\,\mathrm{MeV}$. As pointed out in [4], f_π itself has an ambiguity of order $m_\pi^2/m_\rho^2 \sim 3\%$, the accuracy of the chiral lagrangian parameters. Notice the sign alternation in (18), in agreement with the minimal hadronic ansatz for the Π_V–Π_A correlator, constructed in [7] in the large N_c limit.

Finally, we write down the values of the quark condensate and the parameter m_0^2, obtained from the operators (18):

$$\alpha_s \langle \bar{q} q \rangle^2 \left(1 + c_6 \frac{\alpha_s}{\pi}\right) = (262 \pm 9\,\mathrm{MeV})^6, \qquad (19)$$

$$m_0^2 = -O_8^{V-A}/O_6^{V-A} = 1.1 \pm 0.2\,\mathrm{GeV}^2. \qquad (20)$$

The errors in the RHS are purely experimental: they do not include a possible contribution of the operator O_{12} and higher as well as unknown QCD corrections to the condensates. The factor c_6 is scheme dependent and is left arbitrary in (19). The accuracy of m_0^2 is better than the accuracy of O_8^{V-A} because of the high covariance of O_6 and O_8. Notice the very good agreement of the $D = 10$ condensate (18) obtained from the sum rules with the one estimated in the previous section, (13), within the framework of the factorization hypothesis.

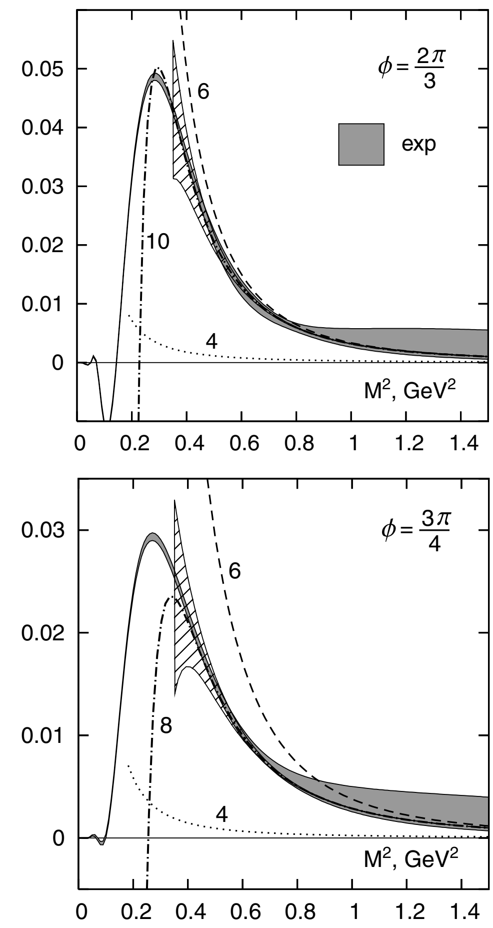

Fig. 4. Imaginary part of the Borel transformation for $\phi = 2\pi/3$ (no O_8) and $\phi = 3\pi/4$ (no O_{10}). The shaded area is the LHS of (15) calculated from the experimental data (with error). The lines display the operator series in the RHS of (15) with condensates equal to the central values of (18). The number nearby each line shows the order of the series; say "8" denotes the contribution $O_4 + O_6 + O_8$. The grid shows a possible contribution of the operator O_{12} within the limits $|O_{12}^{V-A}| < 3 \times 10^{-3}\,\mathrm{GeV}^{12}$

4 Conclusion

We have performed the analysis of the V–A spectral functions, obtained from hadronic τ-decay channels, with the help of the Borel sum rules. The values of the condensates of dimension $D = 6, 8, 10$ were found, (18), by fitting the theoretical curves of the Borel transform to the experimental ones within its error bands. The major contribution to these condensates comes from the 4-quark operators. Its contribution to the current correlators was calculated and their size was estimated by means of the factorization hypothesis. The estimated value of the $D = 10$ condensate (13) is found to be in good agreement with the fit result (18), which demonstrates the validity of the OPE approach in quantum chromodynamics.

Our results are based on several assumptions; in particular, the factorization (vacuum insertion) hypothesis. There is a statement in the literature [26] that the factorization hypothesis underestimates the quartic condensates

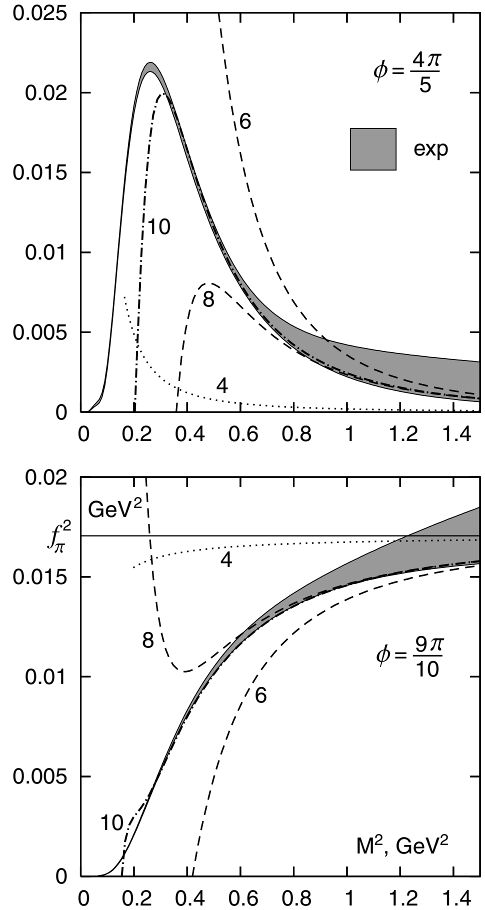

Fig. 5. Imaginary part of the Borel transformation (15) at $\phi = 4\pi/5$ and real part (14) at $\phi = 9\pi/10$. The operator O_{12}^{V-A} vanishes in these sum rules

by a factor ~ 3. This conclusion is based on the comparison of the quark condensate obtained from the $D = 6$ operator in ρ-meson (vector) sum rules with the one calculated from the low energy GMOR theorem. (Our result (19) is also larger than the GMOR condensate for reasonable theoretical parameters.) However this comparison has many other sources of error, such as a scale-scheme ambiguity, high-order QCD corrections, light quark masses, corrections from the chiral lagrangian, etc. The accuracy of the factorization hypothesis can be of the same order as the ambiguity of the factorization of the $D = 8$ operators on the level of $O(N_c^{-2})$-terms, as demonstrated in [4]. A more careful statement about the validity of the factorization hypothesis could be obtained by evaluating the contribution of the meson states to the 4-quark condensates.

A second objection may concern the rather low value of the Borel mass $M_1^2 = 0.4\,\mathrm{GeV}^2$ used in our fit (16). Indeed, the typical scale where perturbative results for the current correlators are confirmed is $Q^2 \gtrsim 1\,\mathrm{GeV}^2$. But our result for the $D = 10$ operator demonstrates a rather low (power-like) growth of the operators $|O_{2n+2}/O_{2n}| \sim m_0^2$ in the V–A channel. If the operators grow as $|O_{2n}| \sim m_0^{2n}$, then the Borel series behaves as $\mathrm{e}^{-m_0^2/M^2}$. The contribution of the $n+1$-term in the exponent is small for $M^2 \gg m_0^2/n$. So for $n = 5$ the minimal scale $M_1^2 = 0.4\,\mathrm{GeV}^2$ seems reasonable. For a faster growth of the operator series this choice could be inappropriate. For instance, if one plots the Borel transformation versus M^2 with the condensate values obtained in [9], the divergence of the operator series will be obvious already at $M^2 \approx 0.7\,\mathrm{GeV}^2$. However, it should be mentioned that the $D = 10$ condensate obtained there exceeds our value (13) by an order of magnitude. It seems unlikely that one could explain such a discrepancy by the inaccuracy of the factorization. All these assumptions can be confirmed or disproved only within a non-perturbative approach.

We have neglected the logarithmic terms $\sim \ln Q^2/Q^{2n}$ in the OPE series (3). Such a contribution from the α_s correction to the $D = 6$ condensate (5) has a small numerical factor; its discontinuity along the real axis $Q^2 = -s < 0$ is too small to compare with the spectral function $v_1(s) - a_1(s)$. For this reason it would be interesting to calculate the α_s^2 correction to the $D = 6$ V–A operator and the α_s correction to the operator O_8^{V-A} and include them in the sum rule analysis.

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Appendix A: 4-quark operators

The calculation of the operator contribution to various current correlators can be performed within the framework of the background field method; see for instance [27]. Here we describe the algorithm, conventions and basic formulae, necessary to calculate the contribution of the 4-quark condensates to the 2-current correlator, which correspond to the third diagram of Fig. 2. We also present here the complete form of the 4-quark operators up to dimension $D = 10$. For definiteness we consider only the vector current correlator; the condensate contribution to the axial current correlator is trivially obtained by the substitution $d \to \gamma^5 d$. The contribution of the 4-quark condensates can be written as

$$\begin{aligned}\Pi^V_{\mu\nu}(q) = &-\frac{\mathrm{i}g^2}{4}\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{e}^{\mathrm{i}qx} \\ &\times\Big\langle D^{ab}_{\alpha\beta}(y,z)\,[\bar{u}(x)\gamma_\mu S(x,y)\gamma_\alpha\lambda^a d(y) \\ &\quad+\bar{u}(y)\gamma_\alpha\lambda^a S(y,x)\gamma_\mu d(x)] \\ &\times\left[\bar{d}(0)\gamma_\nu S(0,z)\gamma_\beta\lambda^b u(z)\right. \\ &\quad\left.+\bar{d}(z)\gamma_\beta\lambda^b S(z,0)\gamma_\nu u(0)\right]\Big\rangle. \end{aligned} \tag{A.1}$$

Here $S(x,y) = \langle Tq(x)\bar{q}(y)\rangle$ is the quark Green function and $D^{ab}_{\mu\nu}(x,y) = \left\langle Ta^a_\mu(x)a^b_\nu(y)\right\rangle$ is the gluon Green function in the background gluon field $A_\mu \to A_\mu + a_\mu$. They obey the equations

$$\mathrm{i}\hat{D}_x\,S(x,y) = \mathrm{i}\delta^4(x-y), \tag{A.2}$$

$$\left[D_x^2 g_{\mu\alpha} + 2G_{\mu\alpha}(x)\right]^{ac} D^{cb}_{\alpha\nu}(x,y) = \mathrm{i}\delta^{ab} g_{\mu\nu}\delta^4(x-y), \tag{A.3}$$

where $g_{\mu\nu} = (+,-,-,-)$ is the Minkowski metric. The quarks are massless, the gluon Green function is taken in the Feynman gauge. The covariant derivative and the gluon field strength tensor in the fundamental representation (A.2) are defined as follows:

$$D_\mu = \partial_\mu - \mathrm{i}A_\mu, \quad A_\mu = \frac{g}{2}\lambda^a A^a_\mu,$$
$$G_{\mu\nu} = \mathrm{i}[D_\mu, D_\nu] = \frac{g}{2}\lambda^a G^a_{\mu\nu}, \tag{A.4}$$

where the λ^a are Gell-Mann matrices, $\mathrm{tr}\left(\lambda^a\lambda^b\right) = 2\delta^{ab}$, $[\lambda^a,\lambda^b] = \mathrm{i}f^{abc}\lambda^c$. We shall also use an additional compact notation for these objects in the adjoint representation (A.3):

$$D^{ab}_\mu = \partial_\mu\delta^{ab} + A^{ab}_\mu, \quad A^{ab}_\mu = \frac{g}{2}f^{acb}A^c_\mu, \quad G^{ab}_{\mu\nu} = \frac{g}{2}f^{acb}G^c_{\mu\nu} \tag{A.5}$$

It is convenient to perform a partial Fourier transformation of the Green functions:

$$S(x,y) = \int \frac{\mathrm{d}^4q}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}q(x-y)}\tilde{S}(q,y),$$
$$D^{ab}_{\mu\nu}(x,y) = \int \frac{\mathrm{d}^4q}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}q(x-y)}\tilde{D}^{ab}_{\mu\nu}(q,y). \tag{A.6}$$

Then one can write down the solution of (A.2) and (A.3) as series in powers of the background field A:

$$\tilde{S}(q,y) = S_0(q)\sum_{n=0}^{\infty}\left[\mathrm{i}\hat{A}\left(\hat{x}\right)S_0(q)\right]^n, \tag{A.7}$$

$$\tilde{D}^{ab}_{\mu\nu}(q,y) = \left\{D_0(q)\sum_{n=0}^{\infty}\left[\mathrm{i}R\left(q,\hat{x}\right)D_0(q)\right]^n\right\}^{ab}_{\mu\nu}, \tag{A.8}$$

where

$$S_0(q) = \frac{\mathrm{i}\hat{q}}{q^2}, \quad D_0(q) = -\frac{\mathrm{i}}{q^2}$$

are free propagators, $\hat{x} = y - \mathrm{i}\overrightarrow{\partial}$, the derivative $\overrightarrow{\partial} = \partial/\partial q$ acts on everything from the right as $[\overrightarrow{\partial}_\mu, q_\nu] = g_{\mu\nu}$; R is the following matrix operator:

$$R^{ab}_{\mu\nu}\left(q,\hat{x}\right)$$
$$= \left[-\mathrm{i}q_\alpha A^{ab}_\alpha\left(\hat{x}\right) - \mathrm{i}A^{ab}_\alpha\left(\hat{x}\right)q_\alpha + A^{ac}_\alpha\left(\hat{x}\right)A^{cb}_\alpha\left(\hat{x}\right)\right]g_{\mu\nu}$$
$$+2G^{ab}_{\mu\nu}\left(\hat{x}\right). \tag{A.9}$$

Equations (A.7) and (A.8) can be evaluated in a gauge covariant way in the fixed point gauge $x^\mu A_\mu(x) = 0$, where

$$A_\mu(x) = -\int_0^1 \mathrm{d}\alpha\,\alpha x^\nu G_{\mu\nu}(\alpha x) \tag{A.10}$$
$$= -x^\nu\sum_{n=0}^{\infty}\frac{x^{\alpha_1}\dots x^{\alpha_n}}{(n+2)n!}D_{\alpha_1}\dots D_{\alpha_n}G_{\mu\nu}(0),$$

$$q(x) = \sum_{n=0}^{\infty}\frac{x^{\alpha_1}\dots x^{\alpha_n}}{n!}D_{\alpha_1}\dots D_{\alpha_n}q(0). \tag{A.11}$$

In order to compute the propagators $\tilde{S}$, $\tilde{D}$ for any fixed order n, one has to substitute (A.10) into (A.7) and (A.8), move all the derivatives $\overrightarrow{\partial}$ to the right and then leave only the terms without $\overrightarrow{\partial}$.

The 4-quark condensate contribution (A.1) can be written in terms of the propagators $\tilde{S}$, $\tilde{D}$ as follows:

$$\begin{aligned}
&\Pi^V_{\mu\nu}(q)\\
&= -\frac{\mathrm{i}g^2}{4}\left[\bar{u}\left(-\mathrm{i}\overrightarrow{\partial}\right)\gamma_\mu\tilde{S}\left(q,-\mathrm{i}\overrightarrow{\partial}\right)\gamma_\alpha\lambda^a d\left(-\mathrm{i}\overrightarrow{\partial}\right)\right.\\
&\quad\times\tilde{D}^{ab}_{\alpha\beta}\left(q,-\mathrm{i}\overrightarrow{\partial}\right)X^b_{\nu\beta}(q) + X^b_{\nu\beta}(q)\,\tilde{D}^{ba}_{\beta\alpha}\left(-q,-\mathrm{i}\overleftarrow{\partial}\right)\\
&\quad\left.\times\,\bar{u}\left(-\mathrm{i}\overleftarrow{\partial}\right)\gamma_\alpha\lambda^a\tilde{S}\left(-q,-\mathrm{i}\overleftarrow{\partial}\right)\gamma_\mu d\left(-\mathrm{i}\overleftarrow{\partial}\right)\right],
\end{aligned} \tag{A.12}$$

where

$$\begin{aligned}
X^b_{\nu\beta}(q) &= \bar{d}\left(-\mathrm{i}\overrightarrow{\partial}\right)\gamma_\beta\lambda^b\tilde{S}(q,0)\gamma_\nu u(0)\\
&\quad+\bar{d}(0)\gamma_\nu\tilde{S}\left(-q,-\mathrm{i}\overleftarrow{\partial}\right)\gamma_\beta\lambda^b u\left(-\mathrm{i}\overleftarrow{\partial}\right).
\end{aligned} \tag{A.13}$$

In the functions $\tilde{S}(q,y)$ and $\tilde{D}(q,y)$ the derivatives $\overrightarrow{\partial}$, $\overleftarrow{\partial}$ over the momentum q always stand on the right from any function of q: $\dots q\dots\partial$. The derivatives inside (A.13) do not act on anything outside $X^b_{\nu\beta}$. After these derivatives are evaluated, we compute the transverse part $\Pi^{(1)} = -\Pi_{\mu\mu}/(3q^2)$ defined according to (1). (We also checked that the longitudinal part vanishes, $\Pi^{(0)} = 0$.) And finally, to separate out the Lorentz invariant condensates, we average $\Pi^{(1)}$ over the directions of the vector q_μ according to

$$\overline{q_{\mu_1}\dots q_{\mu_{2n}}} = 2\frac{(2n-1)!!}{(2n+2)!!}(q^2)^n g_{(\mu_1\mu_2}\dots g_{\mu_{2n-1}\mu_{2n})},$$
$$\overline{q_{\mu_1}\dots q_{\mu_{2n+1}}} = 0, \tag{A.14}$$

where $(\mu_1\dots\mu_n)$ denotes the usual index symmetrization with weight $1/n!$. All these calculations were performed by computer.

The most time-consuming part of the calculation is to reduce the large number of terms in the final result to a minimal number of independent structures. For this purpose we employ the quark equation of motion $\hat{D}u = \hat{D}d = 0$ and the "integration by part" identity $\langle A(D_\mu B)\rangle = -\langle(D_\mu A)B\rangle$ (the vacuum average of the total derivative is zero $\langle\partial_x O(x)\rangle = \partial_x\langle O(x)\rangle = \partial_x\langle O(0)\rangle = 0$ for any gauge invariant operator $O(x)$). It allows one to bring the operators to obviously hermitean (real) form, which provides an additional verification of the result.

In order to write down the 4-quark condensates in a compact form, we introduce here the following bilinear quark structures of increasing dimension D:

$$
\begin{aligned}
3D:\quad & A_\alpha = \bar d\lambda\gamma^5\gamma_\alpha u,\\
4D:\quad & B^{(1)}_{\alpha\beta} = \mathrm{i}\left(\bar d\lambda\gamma_\alpha u_\beta - \bar d_\beta\lambda\gamma_\alpha u\right),\\
& B^{(2)}_{\alpha\beta} = \bar d\lambda\gamma^5\gamma_\alpha u_\beta + \bar d_\beta\lambda\gamma^5\gamma_\alpha u,\\
5D:\quad & C^{(1)}_{\alpha\beta\gamma} = \mathrm{i}\left(\bar d\lambda\gamma_\alpha u_{\beta\gamma} - \bar d_{\beta\gamma}\lambda\gamma_\alpha u\right),\\
& C^{(2)}_{\alpha\beta\gamma} = \bar d\lambda\gamma^5\gamma_\alpha u_{\beta\gamma} + \bar d_{\beta\gamma}\lambda\gamma^5\gamma_\alpha u,\\
& C^{(3)}_{\alpha\beta\gamma} = \bar d_\alpha\lambda\gamma^5\gamma_\beta u_\gamma + \bar d_\gamma\lambda\gamma^5\gamma_\beta u_\alpha,\\
& C^{(4)}_{\alpha\beta\gamma} = \bar d\{\lambda, G_{\alpha\beta}\}\gamma_\gamma u,\\
6D:\quad & E^{(1)}_{\alpha\beta\gamma\delta} = \bar d\{\lambda, G_{\alpha\beta}\}\gamma_\gamma u_\delta + \bar d_\delta\{\lambda, G_{\alpha\beta}\}\gamma_\gamma u,\\
& E^{(2)}_{\alpha\beta\gamma\delta} = \bar d\{\lambda, \tilde G_{\alpha\beta}\}\gamma_\gamma u_\delta + \bar d_\delta\{\lambda, \tilde G_{\alpha\beta}\}\gamma_\gamma u,\\
& E^{(3)}_{\alpha\beta\gamma\delta} = \mathrm{i}\left(\bar d\{\lambda, G_{\alpha\beta}\}\gamma^5\gamma_\gamma u_\delta - \bar d_\delta\{\lambda, G_{\alpha\beta}\}\gamma^5\gamma_\gamma u\right),\\
& E^{(4)}_{\alpha\beta\gamma\delta} = \mathrm{i}\left(\bar d\{\lambda, \tilde G_{\alpha\beta}\}\gamma^5\gamma_\gamma u_\delta - \bar d_\delta\{\lambda, \tilde G_{\alpha\beta}\}\gamma^5\gamma_\gamma u\right),\\
& E^{(5)}_{\alpha\beta\gamma\delta} = \bar d\{\lambda, G_{\beta\gamma;\,\alpha}\}\gamma_\delta u,\\
& E^{(6)}_{\alpha\beta\gamma\delta} = \bar d\{\lambda, \tilde G_{\beta\gamma;\,\alpha}\}\gamma_\delta u,\\
7D:\quad & F^{(1)}_{\alpha\beta\gamma} = \mathrm{i}\left(\bar d\{\lambda, J_\alpha\}\gamma^5\gamma_\beta u_\gamma - \bar d_\gamma\{\lambda, J_\alpha\}\gamma^5\gamma_\beta u\right),\\
& F^{(2)}_{\alpha\beta\gamma} = \bar d\{\lambda, \{G_{\alpha\delta}, G_{\delta\beta}\}\}\gamma^5\gamma_\gamma u,\\
& F^{(3)}_{\alpha\beta\gamma} = \mathrm{i}\bar d\{\lambda, [G_{\alpha\delta}, \tilde G_{\delta\beta}]\}\gamma_\gamma\ u,
\end{aligned}
\tag{A.15}
$$

where $u_\alpha = D_\alpha u$, $u_{\alpha\beta} = D_{(\alpha}D_{\beta)}u \equiv \frac{1}{2}(D_\alpha D_\beta + D_\beta D_\alpha)u$, $G_{\beta\gamma;\,\alpha} = D_\alpha G_{\beta\gamma}$, $J_\alpha = D_\beta G_{\alpha\beta}$, $[A,B] = AB - BA$, $\{A,B\} = AB + BA$. The dual tensor is defined by $\tilde G_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}G_{\mu\nu}$, $\varepsilon^{0123} = 1$ and $\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$. The values G, $\tilde G$, J in (A.15) are in the fundamental representation. All bilinear structures belong to the adjoint representation of the gauge group, and the gauge index of the Gell-Mann matrices λ is omitted. We denote conjugated structures by overlined letters, which are simply obtained by the replacement $u \rightleftarrows d$, for instance $\bar A_\alpha \equiv A^\dagger_\alpha = \bar u\lambda\gamma^5\gamma_\alpha\ d$.

The 4-quark condensates of dimension $D = 6, 8, 10$ are

$$
O_6^V = -2\pi\alpha_\mathrm{s}\left\langle \bar A_\alpha A_\alpha\right\rangle, \tag{A.16}
$$

$$
\begin{aligned}
O_8^V = &\ \frac{2\pi\alpha_\mathrm{s}}{9}\\
&\times\Big\langle -4\bar B^{(1)}_{\alpha\beta}B^{(1)}_{\alpha\beta} - \bar B^{(2)}_{\alpha\beta}B^{(2)}_{\alpha\beta}\\
&-4\bar C^{(3)}_{\beta\alpha\beta}A_\alpha - 4\bar A_\alpha C^{(3)}_{\beta\alpha\beta} + 12\bar A_\alpha G_{\alpha\beta}A_\beta\Big\rangle,
\end{aligned}
\tag{A.17}
$$

$$
\begin{aligned}
O_{10}^V = &\ \frac{\pi\alpha_\mathrm{s}}{9}\Big\langle 25\bar C^{(1)}_{\alpha\beta\gamma}C^{(1)}_{\alpha\beta\gamma} - 5\bar C^{(2)}_{\alpha\beta\gamma}C^{(2)}_{\alpha\beta\gamma} - 10\bar C^{(3)}_{\alpha\beta\alpha}C^{(3)}_{\gamma\beta\gamma}\\
&-19\bar C^{(4)}_{\alpha\beta\beta}C^{(4)}_{\alpha\gamma\gamma} - \frac{15}{4}\bar C^{(4)}_{\alpha\beta\gamma}C^{(4)}_{\alpha\beta\gamma} - 8\bar C^{(4)}_{\alpha\beta\gamma}C^{(4)}_{\beta\gamma\alpha}\\
&-2\bar B^{(1)}_{\alpha\beta}G_{\beta\gamma}B^{(1)}_{\alpha\gamma} - 66\bar B^{(2)}_{\alpha\beta}G_{\alpha\gamma}B^{(2)}_{\gamma\beta}\\
&+\bar A_\alpha\Big(8J_{[\alpha;\,\beta]} - 3G_{\alpha\gamma}G_{\gamma\beta} + 19G_{\beta\gamma}G_{\gamma\alpha}\Big)A_\beta\\
&+\frac{33}{4}\bar A_\alpha G_{\beta\gamma}G_{\beta\gamma}A_\alpha\\
&+\bar B^{(1)}_{\alpha\beta}\Big(E^{(1)}_{\beta\gamma\alpha\gamma} + \frac{5}{2}E^{(4)}_{\alpha\gamma\gamma\beta} - \frac{7}{2}E^{(5)}_{\gamma\beta\gamma\alpha} - 28\tilde G_{\beta\gamma}B^{(2)}_{\alpha\gamma}\\
&+\frac{21}{2}\tilde G_{\beta\gamma;\,\alpha}A_\gamma\Big) + \Big(\bar E^{(1)}_{\beta\gamma\alpha\gamma} + \frac{5}{2}\bar E^{(4)}_{\alpha\gamma\gamma\beta} - \frac{7}{2}\bar E^{(5)}_{\gamma\beta\gamma\alpha}\\
&+28\bar B^{(2)}_{\alpha\gamma}\tilde G_{\beta\gamma} - \frac{21}{2}\bar A_\gamma\tilde G_{\beta\gamma;\,\alpha}\Big)B^{(1)}_{\alpha\beta}\\
&+\bar B^{(2)}_{\alpha\beta}\Big(-\frac{11}{2}E^{(2)}_{\alpha\gamma\gamma\beta} + \frac{15}{4}E^{(2)}_{\beta\gamma\alpha\gamma} + \frac{5}{2}E^{(3)}_{\alpha\gamma\beta\gamma} + 5E^{(3)}_{\beta\gamma\alpha\gamma}\\
&+\frac{1}{2}E^{(6)}_{\alpha\beta\gamma\gamma} + \frac{3}{2}E^{(6)}_{\beta\alpha\gamma\gamma} - 4E^{(6)}_{\gamma\alpha\beta\gamma}\\
&-\frac{17}{2}G_{\alpha\beta;\,\gamma}A_\gamma - 11J_\beta A_\alpha\Big)\\
&+\Big(-\frac{11}{2}\bar E^{(2)}_{\alpha\gamma\gamma\beta} + \frac{15}{4}\bar E^{(2)}_{\beta\gamma\alpha\gamma}\\
&+\frac{5}{2}\bar E^{(3)}_{\alpha\gamma\beta\gamma} + 5\bar E^{(3)}_{\beta\gamma\alpha\gamma} + \frac{1}{2}\bar E^{(6)}_{\alpha\beta\gamma\gamma} + \frac{3}{2}\bar E^{(6)}_{\beta\alpha\gamma\gamma}\\
&-4\bar E^{(6)}_{\gamma\alpha\beta\gamma} + \frac{17}{2}\bar A_\gamma G_{\alpha\beta;\,\gamma} + 11\bar A_\alpha J_\beta\Big)B^{(2)}_{\alpha\beta}\\
&+\bar A_\alpha\Big(-3F^{(1)}_{\beta\alpha\beta} + F^{(1)}_{\beta\beta\alpha} + 2F^{(2)}_{\alpha\beta\beta} - \frac{1}{2}F^{(2)}_{\beta\beta\alpha} + F^{(3)}_{(\alpha\beta)\beta}\\
&-\frac{45}{2}\tilde G_{\alpha\beta}C^{(4)}_{\beta\gamma\gamma} + \frac{35}{2}\tilde G_{\beta\gamma}C^{(4)}_{\alpha\beta\gamma} + \frac{75}{8}\tilde G_{\beta\gamma}C^{(4)}_{\beta\gamma\alpha}\Big)\\
&+\Big(-3\bar F^{(1)}_{\beta\alpha\beta} + \bar F^{(1)}_{\beta\beta\alpha} + 2\bar F^{(2)}_{\alpha\beta\beta} - \frac{1}{2}\bar F^{(2)}_{\beta\beta\alpha} + \bar F^{(3)}_{(\alpha\beta)\beta}\\
&+\frac{45}{2}\bar C^{(4)}_{\beta\gamma\gamma}\tilde G_{\alpha\beta} - \frac{35}{2}\bar C^{(4)}_{\alpha\beta\gamma}\tilde G_{\beta\gamma} - \frac{75}{8}\bar C^{(4)}_{\beta\gamma\alpha}\tilde G_{\beta\gamma}\Big)A_\alpha\Big\rangle.
\end{aligned}
\tag{A.18}
$$

In (A.17) and (A.18) the field strengths G, $\tilde G$, J are in the adjoint representation $G^{ab}_{\alpha\beta} = \frac{g}{2}f^{acb}G^c_{\alpha\beta}$ etc.; gauge indices are omitted; say $\bar A_\alpha G_{\beta\gamma}G_{\beta\gamma}A_\alpha$ denotes $\bar A^a_\alpha G^{ab}_{\beta\gamma}G^{bc}_{\beta\gamma}A^c_\alpha$. The operator O_8^V (A.17) can be easily brought to the form obtained in [4,19].

Appendix B: Factorization of 4-quark condensates

At first let us recall how the factorization (vacuum insertion) works for the $D = 6$ operators. This is illustrated by the following equation:

$$
\langle(\bar u\lambda\Gamma_1 d)(\bar d\lambda\Gamma_2 u)\rangle = -2\,C_N\ \mathrm{tr}\left[\langle u\otimes\bar u\rangle\Gamma_1\langle d\otimes\bar d\rangle\Gamma_2\right], \tag{B.1}
$$

where Γ_i are some Dirac matrices, $C_N = 1 - 1/N_c^2$, N_c is the color number, kept arbitrary here. In (B.1) the notation $\langle q\otimes\bar q\rangle$ denotes a 4×4 matrix in spinor space; the color indices are contracted. It is proportional to the quark condensate:

$$
\langle q\otimes\bar q\rangle = -\frac{1}{4}\langle\bar q q\rangle. \tag{B.2}
$$

The result of the factorization is well known:

$$O_6^V = -4\pi\alpha_s C_N \langle\bar{u}u\rangle\langle\bar{d}d\,\rangle. \tag{B.3}$$

(In the vector sum rules one also accounts for the additional operator $\langle\bar{q}\gamma_\alpha J_\alpha q\rangle$, which takes the 4-quark form when the gluon equation of motion is applied. Such an operator comes from the 2-quark diagram, so it cancels in the V–A correlators.)

The factorization procedure becomes ambiguous at the level of $D = 8$ 4-quark condensates. As shown in [4], different ways of factorization give different terms $\sim 1/N_c^2$. For definiteness, here we follow the following factorization scheme. At first we replace the field strength by the derivatives as follows: $G_{\mu\nu} = \mathrm{i}[D_\mu, D_\nu]$ for the fundamental representation and $G_{\mu\nu}^{ab} = [D_\mu, D_\nu]^{ab}$ for the adjoint one. Then we apply (B.1), where the quark wave functions $u, \bar{u}$ and $d, \bar{d}$ may carry some derivatives. Finally, the quark matrices $\langle\ldots\rangle$ with derivatives are expressed in terms of the condensates as

$$\langle D_\alpha q \otimes \bar{q}\rangle = 0, \tag{B.4}$$

$$\langle D_\alpha D_\beta q \otimes \bar{q}\rangle = -\frac{1}{32}\left(g_{\alpha\beta} + \frac{1}{3}\gamma_{\alpha\beta}\right)\mathrm{i}\langle\bar{q}\hat{G}q\rangle,$$

where $\gamma_{\alpha\beta} = \gamma_{[\alpha}\gamma_{\beta]} = \frac{1}{2}(\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha)$, $\hat{G} = \gamma_{\alpha\beta}G_{\alpha\beta}$. The result for the $D = 8$ condensate is

$$O_8^V = -2\pi\alpha_s C_N \left[\langle\bar{u}u\rangle\,\mathrm{i}\langle\bar{d}\hat{G}d\rangle + \langle\bar{d}d\rangle\,\mathrm{i}\langle\bar{u}\hat{G}u\rangle\right]. \tag{B.5}$$

In the condensate O_{10} one encounters the terms with quarks carrying four derivatives. We average these terms with the help of the following rule:

$$\begin{aligned}
&\langle D_\alpha D_\beta D_\gamma D_\delta q \otimes \bar{q}\rangle \\
&= -\frac{1}{24^2}\left[g_{\alpha\beta}g_{\gamma\delta}(3X_2 + 6X_3 - 2X_4)\right. \\
&\quad + g_{\alpha\gamma}g_{\beta\delta}(6X_1 + 3X_2 + 6X_3 + 4X_4) \\
&\quad + g_{\alpha\delta}g_{\beta\gamma}(12X_1 + 3X_2 + 6X_3 + 4X_4) \\
&\quad + (g_{\alpha\beta}\gamma_{\gamma\delta} + g_{\gamma\delta}\gamma_{\alpha\beta})(X_1 + 2X_2 + 3X_3) \\
&\quad + (g_{\alpha\gamma}\gamma_{\beta\delta} + g_{\beta\delta}\gamma_{\alpha\gamma})(2X_1 + X_2 + 3X_3 + X_4) \\
&\quad + g_{\alpha\delta}\gamma_{\beta\gamma}(X_1 + 2X_2 + X_4) + g_{\beta\gamma}\gamma_{\alpha\delta}(X_1 + X_4) \\
&\quad \left.+ 3\gamma_{\alpha\beta\gamma\delta}X_2\right],
\end{aligned} \tag{B.6}$$

where $\gamma_{\alpha\beta\gamma\delta} = \gamma_{[\alpha}\gamma_\beta\gamma_\gamma\gamma_{\delta]}$, and X_i are 7-dimensional condensates, defined in (10).

Having been applied to the operator (A.18), this procedure gives the following result:

$$\begin{aligned}
O_{10}^V &= \pi\alpha_s C_N \\
&\times\left[\frac{25}{9}\langle\bar{u}\hat{G}u\rangle\langle\bar{d}\hat{G}d\rangle - 4\left(3X_1^u - X_2^u + X_3^u + \frac{7}{6}X_4^u\right)\langle\bar{d}d\rangle\right. \\
&\quad \left.-4\left(3X_1^d - X_2^d + X_3^d + \frac{7}{6}X_4^d\right)\langle\bar{u}u\rangle\right],
\end{aligned} \tag{B.7}$$

where the X_i^q are constructed from the quark of flavor q. The axial condensates can be obtained by the simple replacement $d \to \gamma^5 d$. For all factorized 4-quark operators $O_D^A = -O_D^V$.

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